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An Imcompressible Rayleigh-Taylor Problem in KULL

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The goal of the EZturb mix model in KULL is to predict the turbulent mixing process as it evolves from Rayleigh-Taylor, Richtmyer-Meshkov, or Kelvin-Helmholtz instabilities. In this report we focus on a simple example of the Rayleigh-Taylor instability (which occurs when a heavy fluid lies above a light fluid, and we perturb the interface separating them). It is well known that the late time asymptotic, fully self-similar form for the growth of the mixing zone scales quadratically with time.

In Kull, the EZturb k- ε model [1-2] is tightly coupled to the Lagrange hydro, and so the actual mix model equations we solve when the model is active are:

$$\frac{d\alpha_r}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_M}\frac{\partial}{\partial x_j}\left(\frac{\alpha_r}{\rho}\right)\right)$$

$$\frac{d\alpha_r\rho_r}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_M}\frac{\partial}{\partial x_j}\left(\frac{\alpha_r\rho_r}{\rho}\right)\right)$$

$$\frac{d\rho u_i}{dt} = -\frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{d\alpha_r\rho_r I_r}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_U}\frac{\partial}{\partial x_j}\left(\frac{\alpha_r\rho_r I_r}{\rho}\right)\right) + P_r^I + \delta_{I,Diss}\alpha_r\rho_r\varepsilon$$

$$\frac{d\rho k}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_K}\frac{\partial k}{\partial x_j}\right) - \tau_{ij}S_{ij} - \rho\varepsilon + P$$

$$\frac{d\rho\varepsilon}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_Z}\frac{\partial \varepsilon}{\partial x_j}\right) - C_{1\varepsilon}\tau_{ij}S_{ij}\frac{\varepsilon}{k} - C_{2\varepsilon}\rho\frac{\varepsilon^2}{k} + C_{3\varepsilon}P\frac{\varepsilon}{k} \quad .$$

Here, α_r, ρ_r, and I_r are the volume fraction, thermodynamic density, and specific internal energy (by mass) for material r. S_{ij} is the strain rate tensor, and τ_{ij} is the turbulent shear stress tensor, for which we use the following Boussinesq approximation:

$$\tau_{ij} = \delta_{Iso}\frac{2}{3}\rho k\delta_{ij} - \delta_{Anso}2\mu_t\left(S_{ij} - \frac{\delta_{ij}}{3}\frac{\partial u_k}{\partial x_k}\right) \quad .$$

The turbulent viscosity includes the effects of both shear and buoyancy and takes the form:

$$\mu_t = C_\mu\rho\frac{k^2}{\varepsilon} + C_\omega\rho\frac{k^3}{\varepsilon^2}\sqrt{\omega^2 + \frac{\nabla p\cdot\nabla p}{p\rho} - \frac{\nabla p\cdot\nabla\rho}{\rho^2}\left[1-\Theta(\nabla p\cdot\nabla\rho)\right]} \quad .$$

The unlimited form of the buoyant production term is given by:

$$P = -\frac{\mu_t}{\sigma_\rho\rho^2}\nabla p\cdot\nabla\rho$$

and the way this term manifests itself in the internal energy equation is:

$$P_r^I = -h_r p_r \nabla\cdot\left(\frac{\mu_t}{\sigma_\rho\rho^2}\nabla\rho\right) \quad .$$

The model constants are $C_{1\varepsilon}$, $C_{2\varepsilon}$, $C_{3\varepsilon}$, σ_M, σ_U, σ_K, σ_Z, σ_ρ, C_μ, and C_ω. Also, δ_{Iso}, δ_{Aniso}, and $\delta_{I,diss}$ are on/off switches that can be set to 1 or 0. To simplify the form of the turbulent viscosity and the Reynolds stress, we will set $C_\omega = 0$ and $\delta_{Aniso} = 0$ for this problem.

Given the complexity of the mix model and the number of coupled PDEs we need to solve to compute the growth of the turbulent mixing zone, it is worthwhile to verify a priori whether the EZturb model is consistent with t^2 growth. Let's start by defining three length scales, where the first will be formed using the turbulent diffusivity (ν) and the acceleration (g), the second will be formed using the turbulent kinetic energy (k) and the turbulent dissipation (ε), and the third will be formed using g and t. Dimensional analysis gives:

$$L_1 = \left(\frac{\nu^2}{g}\right)^{1/3}$$

$$L_2 = \left(\frac{k^{3/2}}{\varepsilon}\right)$$

$L_3 = g\,t^2$.

Now assume that $k \sim t^a$, $\varepsilon \sim t^b$, $\nu \sim k^2/\varepsilon$, and that there is constant acceleration. Since fully self-similar behavior demands that all length scales grow at the same rate, we can solve two linear equations to find that a =2 and b=1. Using the fact that ρ is bounded, the pressure gradient is bounded by ρg, and that the gradient operator scales like inverse length (or t^{-2}), we can show that all the terms in the EZturb k-eqn grow linearly in time, and that all the terms in the ε-eqn have a constant behavior in time (provided k and ε have the simple time scalings mentioned above). Therefore, when these equations are integrated in time, we recover the result that the kinetic energy grows quadratically in time, while the dissipation grows linearly in time. Thus, the EZturb model admits t^2 growth as a possible solution.

Consider a simple 1 dimensional RT problem in a box of length L, with the interface separating the heavy fluid from the light fluid located at x = L/2. Hydrostatic pressure balances give $P = P_{int} - \rho_L g x'$ for $-L/2 < x' < 0$ and $P = P_{int} - \rho_H g x'$ for $0 < x' < L/2$ (here, the subscript 'L' denotes light while 'H' denotes heavy). Note that we have also made a linear transformation from x to x', where $0 < x < L$ and $-L/2 < x' < L/2$, so that the location of the interface is now at x' = 0. We will take L = 2 cm, $\rho_H = 2$ g/cm^3, $\rho_L = 1$ g/cm^3, g = 2 cm/sh^2, and $P_{int} = 100$ jerks/cm^3. The heavy and light fluids will be treated as gamma law gases with $\gamma_H = 5/3$ and $\gamma_L = 1.4$. Since we will be running the RT problem in a Lagrangian frame, we will use constant pressure boundary conditions to provide the necessary acceleration. If we were running Eulerian (Lagrangian step plus remap), then we would impose zero velocity at the walls.

We also want to run this problem in the nearly incompressible regime. To see what constraints this places on the initial conditions, consider the definition of the compressibility, τ, which is (1/ρ) dρ/dp. This implies dρ = ρτ dp, and 'incompressible' means that a change in p (when multiplied by the density and compressibility) will only cause a small change in ρ. If we divide both sides by dx and assume an ideal gas for relating density to pressure, then we arrive at the simple result that (1/ρ) dρ/dx = (1/p) dp/dx. Since the pressure gradient for our RT problem scales like ρg, we see that we want to keep ρg/P small. This explains why we are using such a large value for the interface pressure.

In addition to setting thermodynamic quantities like density and pressure, we also need to initialize values for the turbulent kinetic energy and dissipation. To this end, we have specified an initial turbulent kinetic energy of 1.0e-6 cm^2/sh^2 and an initial length scale of 4.0e-6 cm. We then use $\varepsilon_0 = (k_0)^{3/2}/L_0$ to set the value for the dissipation. Note that there is nothing preventing us from specifying energy or length scale as a function of position. We are simply choosing to set very small constant values for the turbulent quantities and allowing them to build up over time through the buoyant production terms. For the model constants and switches, we are using the following values:

$\sigma_M = \sigma_\rho = \sigma_U = \sigma_K = .7$, $\sigma_e = 1.3$, $c_\mu = .09$, $c_{\varepsilon 1} = 1.44$, $c_{\varepsilon 2} = 1.92$, $c_{\varepsilon 3} = 1.0$, $\delta_{iso} = 1$, $\delta_{aniso} = 0$, $\delta_{I,diss} = 1$.

Also, we assume the fluids are initially at rest and we set the artificial viscosity to zero.

Figure 1 shows the results of a convergence study, where we have used uniform zoning on meshes with 50, 100, 200, and 400 zones. For computing the mixing zone widths, we have interpolated to find the x-locations for the 5% and 95% values in volume fraction. First, we can notice that mixing does not start at t=0, but rather it takes time for the turbulent quantities (in particular, the turbulent diffusivity and length scale) to grow in the zones adjacent to the fluid interface. Also, it may seem strange that we are stopping the runs when the width is only approaching 1 cm or half of the box length. In reality, the tails on the volume fraction distribution are so large that the bubbles and spikes are hitting the walls at ~ 5.5 sh. The time step is also extremely small for these calculations due to running in the nearly incompressible limit. For example, the time step for the 100 zone run is ~ 3.0e-4 sh at early time and decreases as more mixing takes place.

To determine whether we are achieving t^2 growth for the mixing zone, it is easier to look at the growth rate, since quadratic growth implies a linear *rate* of growth. Numerical differentiation of an interpolated quantity can be noisy, and so to give the mixing more time to evolve and reach a self-similar state, the box size was doubled for the 50 and 100 zone runs. These results for the time evolution of the growth rate are shown in Figure 2. While the 50 zone run takes a long time to settle into a linear growth rate (approximately 5 sh), the 100 zone run appears to reach linear behavior at about 3 sh.

One can legitimately ask if there is a way to accelerate the onset of growth when we initialize the turbulent length scale to be several orders of magnitude below the grid spacing. Fortunately, the answer is yes, although we will need to slightly modify the form of the buoyant production term. Basically, if we express the buoyant production term in term of k and L variables, then there is a factor of $k^{1/2}$ L that comes from the kinematic viscosity. Clearly, if L is very small, then the production term will also be small, and will remain so until k and ε grow to the point such that the resulting $L = k^{3/2} / \varepsilon$ is of $O(\Delta x)$. To get contributions at early time, we can multiply the production term by the factor $(1+\Delta x/L)$ [3]. Thus, when L is small compared to Δx, the buoyant production term has a significant effect.

Figure 3 shows the results of a convergence study with this extra factor included in the production term. Now, we do not see the long delay before the onset of growth. Also, as we increase the zoning, the widths get smaller, which is more consistent with the fact that the volume fraction profiles are getting steeper under mesh refinement. Figure 4 shows the results of doubling the box length for the 50 and 100 zone calculations, so that they can run longer in time. From this plot, it appears as though we obtain quadratic growth around t=5 sh. Since many people often ask what the value of α is in the asymptotic growth of the mixing zone, we can compute the slope from this figure at late time and divide it by 2 At g. The result is a value of a of .053, which means the α_{bubble} is ~.0265. What is important is not the actual value of a, since a self-similar solution would uniquely define the value in terms of model constants (the subject of future work), but rather that we get the correct temporal scaling.

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References:

[1] M. Ulitsky, *The Barenblatt Turbulent Burst Problem in KULL*, UCRL-TR-212637.

[2] M. Ulitsky, *The Benjamin Shock Tube Problem in KULL*, UCRL-TR-214888.

[3] Discussions with George Zimmerman.

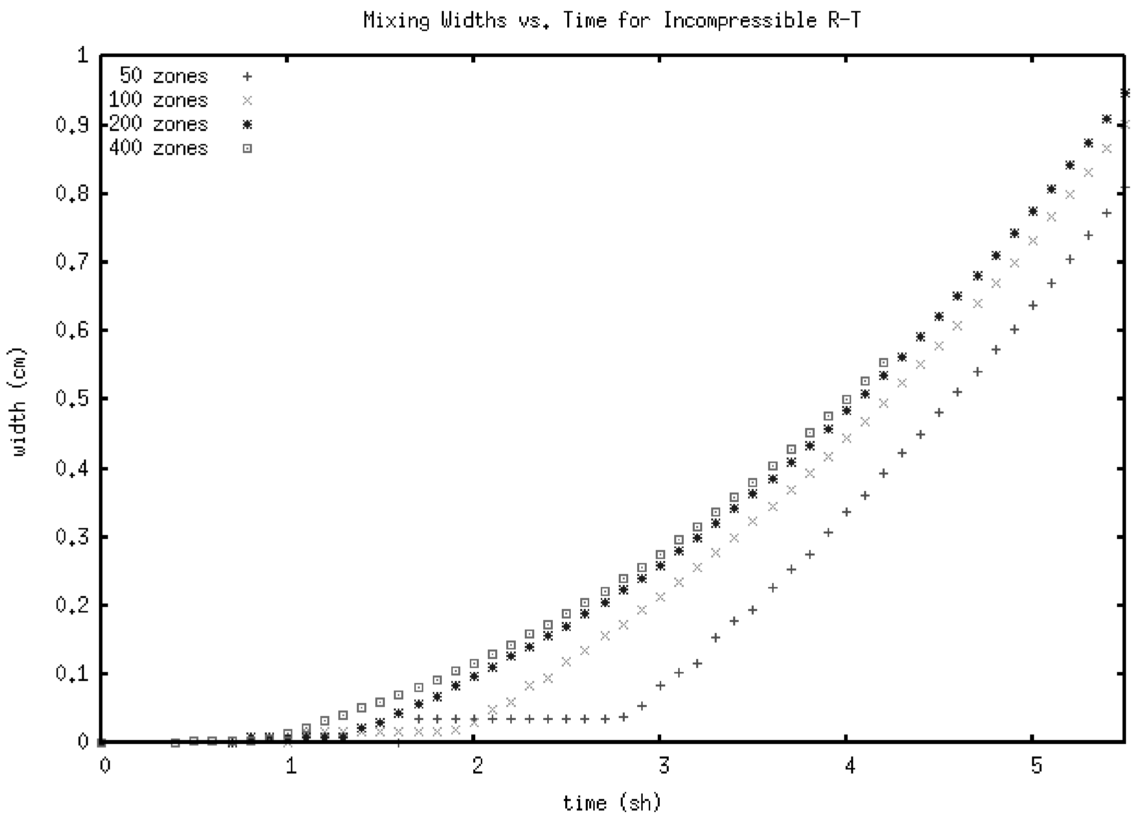
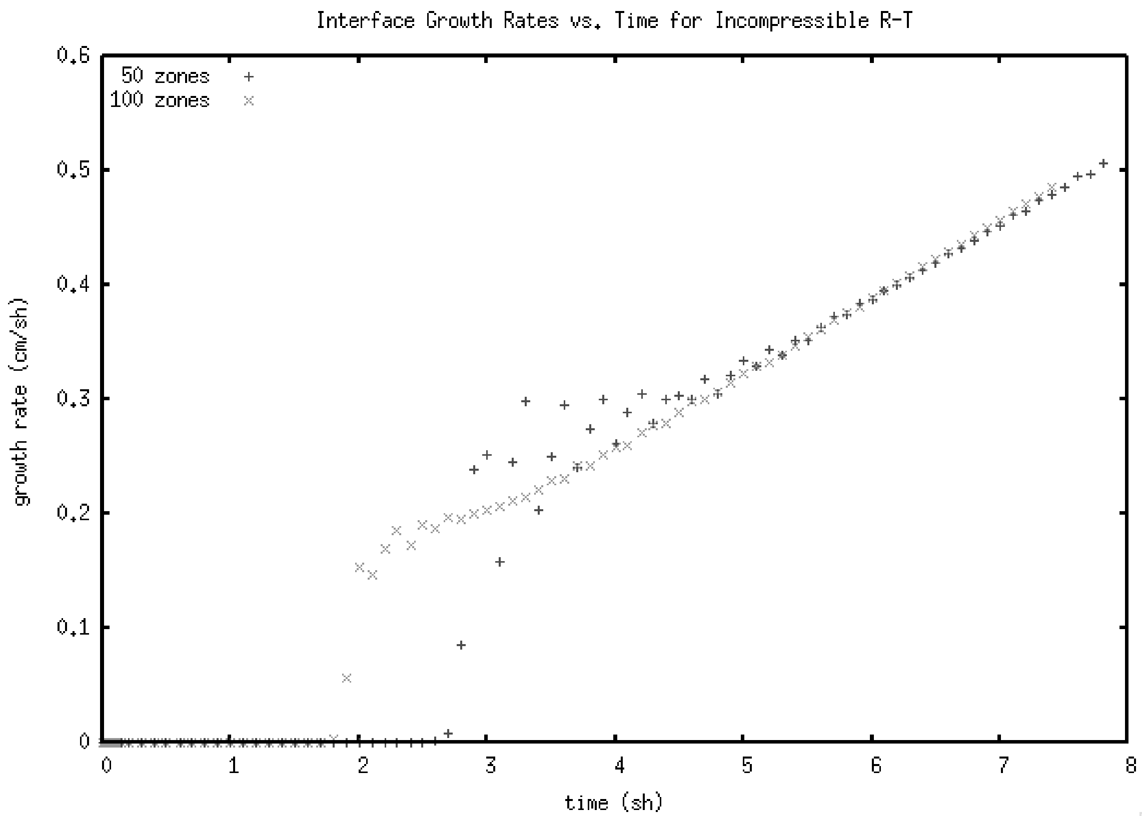


Figure 1



Interface Growth Rates vs. Time for Incompressible R-T
50 zones
100 zones
growth rate (cm/sh)
time (sh)
0
0.1
0.2
0.3
0.4
0.5
0.6
1
2
3
4
5
6
7
8

Figure 2

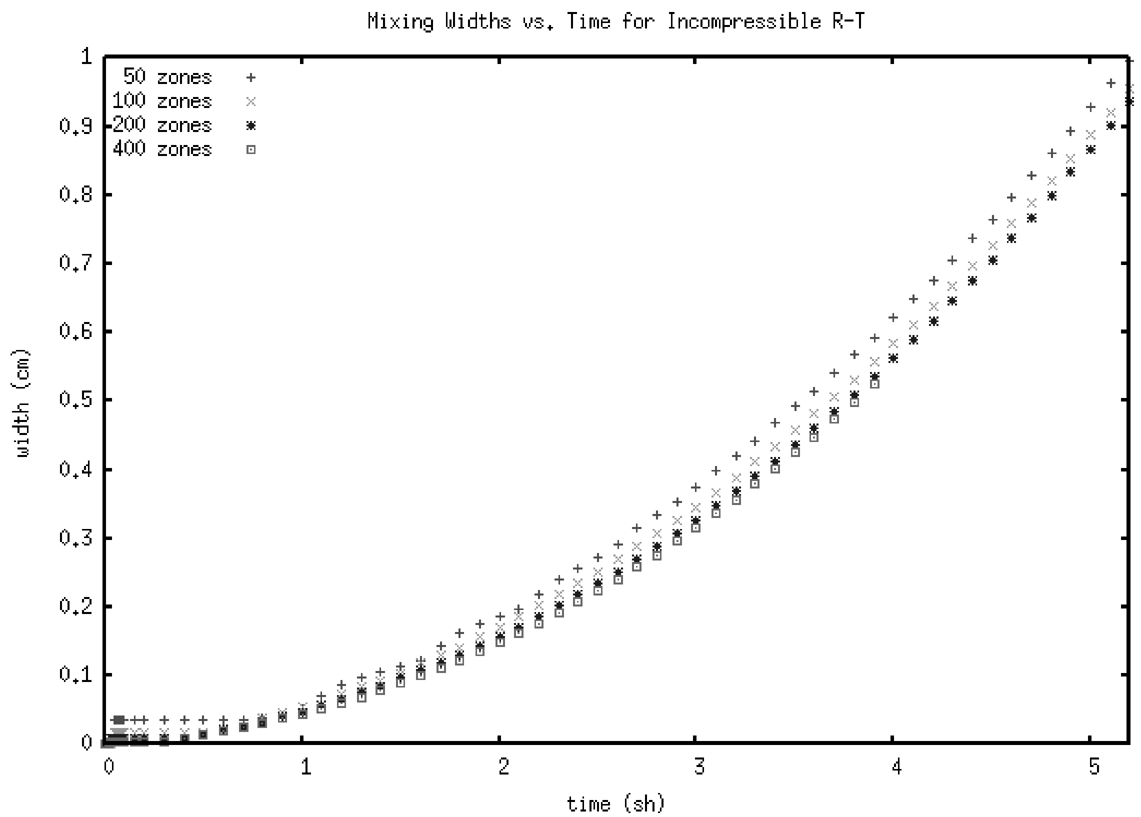
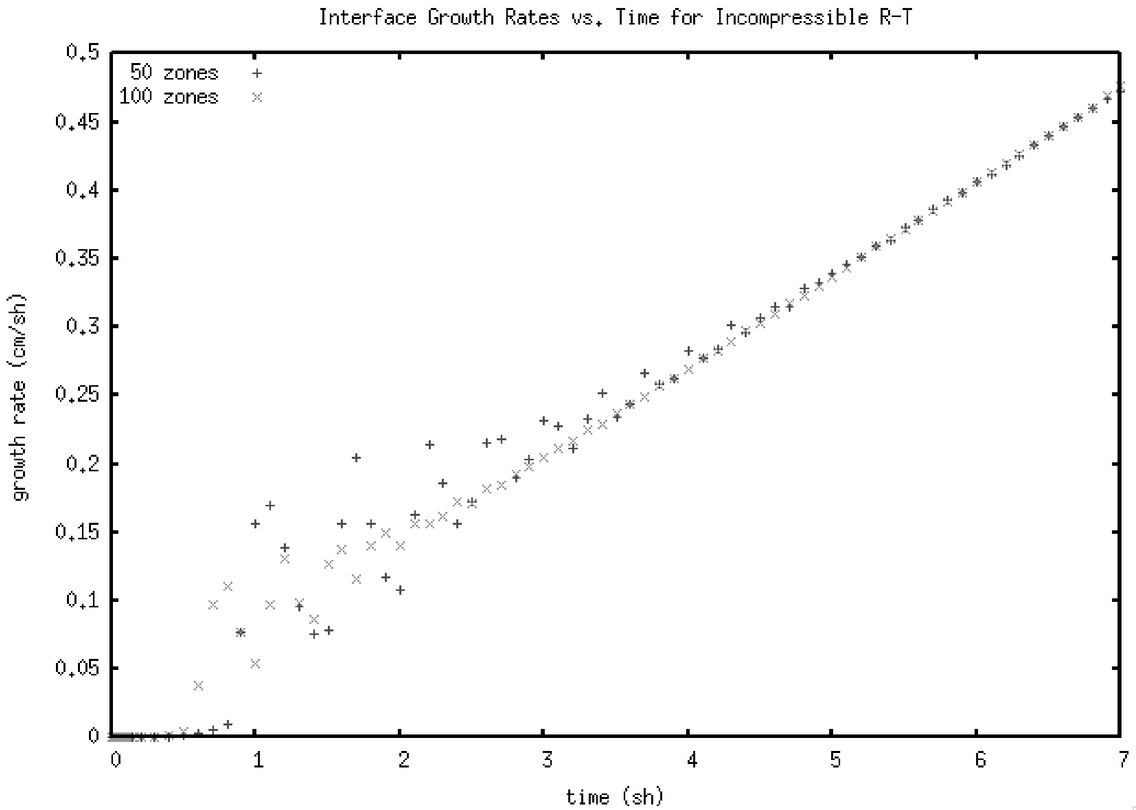

Mixing Widths vs. Time for Incompressible R-T
50 zones
100 zones
200 zones
400 zones
width (cm)
time (sh)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
2
3
4
5

Figure 3



Interface Growth Rates vs. Time for Incompressible R-T
50 zones
100 zones
growth rate (cm/sh)
time (sh)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7

Figure 4